

ENTITIES, IDENTITY AND THE FORMAL STRUCTURE OF QUANTUM MECHANICS

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Abstract

The concept of individuality in quantum mechanics shows radical differences from the one used in classical physics. In particular, it is not possible to consider the fundamental particles described by quantum theory as individual distinguishable objects. In this paper we present arguments in favor of quantum non-individuality, which—in addition to those based on quantum statistics—relate to the Kochen-Specker theorem and the principle of superposition. Then, we analyze the possibility of referring to ‘possible individuals’ instead of ‘actual individuals’, and show that the Modal-Kochen-Specker theorem precludes this interpretational move.

1 Classical physics as a theory of entities

Using the opportunity of this interdisciplinary conference on the subject of *Structure and Identity* we would like to present a paper which discusses the physical and philosophical problems engaged with the interpretation of the formal structure of quantum mechanics in relation to the notions of entity and identity.

Aristotle relates ‘identity’ to the concept of ‘entity’ via his formulation of logic. Plato had designed this concept in order to escape the problem of movement in which Pre-Socratic philosophy had based itself. The principles of Aristotelian logic, namely, *the existence of objects of knowledge*, *the principle of contradiction* and *the principle of identity*, appear thus, as the conditions of possibility to refer to an entity.¹ But as stressed repeatedly by Martin Heidegger, since Plato,² philosophy has thought of Being—the most radical question regarding philosophy—in the very specific terms of an entity, forgetting the question which guided philosophy in the first place. Alfred North Whitehead also referred to this aspect of occidental philosophy mentioning with an ironic glance that: “The safest general characterization of the European philosophical tradition is that it consists of a series of footnotes to Plato” [42]. In relation to physics, the idea of entity played the most radical role in understanding phenomena, so in this very same sense, as

¹See for discussion [40].

²We could add Aristotle as that whom structures the notion of entity beyond philosophy and constructs the logical structure of it.

discussed in [27, 28], we might say that the history of classical physics has been confined —since Plato, but more specifically in relation to physics, since Aristotle— to the history of ‘physical entities’, i.e. particles, waves, fields, etc.

The foundational revolution in mathematics and physics, which took place in between the end of the 19th century and the beginning of the 20th, shacked the very cornerstones of classical thought. Quantum mechanics can be regarded as one of the most definite blows received by the classical world-view. The impossibility to refer to an object, an individual, reflected the limitations of the physics which had been thought until then, a physics which had been founded on the concept of ‘entity’ and the logical structure of Aristotle. Expressing the idea that one needs to apply the principle of identity to every entity, Heidegger states in *Der Staz der Identität* that the principle is a law of Being which states that identity —the unit with itself— pertains to each entity as such. This principle —in the form ‘ $a = a$ ’— belongs to the axioms of set theory, which is in turn in the basis of the mathematics in which physical theories are axiomatized.

We will analyze the compatibility between the notions of entity and identity and the formal structure of quantum mechanics. To do so, we assume that a certain description is determined by a set of concepts which interrelate and that a concept is defined by its inter-relations to other concepts. Identity and individuality are two main constituters of the notion of entity. In this respect, many discussions tend to ‘dissect’ the concept of entity, as if, when tearing apart the concept from its constituents they would be still allowed to talk about the same concept.³ But the notion of individual and that of properties are *interdefined* notions. The individual entity is constituted by the notions of existence, identity and contradiction (of properties obeying classical logic), these are notions which interrelate and determine the concept of entity. The fall of one of them causes the whole architectonic to loose its foundation and crumble down into pieces.

We shall firstly review the standard analysis of quantum non-individuality of indistinguishable particles via the quantum statistics and comment an alternative approach to deal with them from quaset theory [4, 21], a landscape where the logical principle of identity has restricted applicability. Then we present other challenges to individuality, namely those posed by the Kochen-Specker theorem and the superposition principle. We also link the discussion on quantum non-individuality to the interpretation of the Fock-space formalism, and relate it to the philosophical problem of the one and the many. Finally we discuss, within the modal scheme if it is achievable to retain, instead of the notion of ‘actual individual’, at least the notion of ‘possible individual’.

2 Individuality in quantum mechanics

Since the origins of quantum mechanics, problems appeared when attempts were made to interpret its formalism and experiments. In particular, a major problem was found in relation to the individuality of quantum systems. An expression of this problem can be found in [34], where Erwin Schrödinger states that: “[...] we have [...] been compelled to dismiss the idea that [...] a particle is an individual entity which retains its ‘sameness’ forever. Quite the contrary, we are now obliged to assert that the ultimate constituents of matter have no ‘sameness’ at all.” And continues: “I beg to emphasize this and I beg you to believe it: It is not a question of our being able to ascertain the identity in some instances and not being able to do so in others. It is beyond doubt that the question of ‘sameness’, of identity, really and truly has no meaning.” Schrödinger extended his assertions in different works, for example:

“I mean this: that the elementary particle is not an individual; it cannot be identified, it lacks ‘sameness’. The fact is known to every physicist, but is rarely

³This is discussed more deeply in [29].

given any prominence in surveys readable by nonspecialists. In technical language it is covered by saying that the particles ‘obey’ a newfangled statistics, either Einstein-Bose or Fermi-Dirac statistics. [...] The implication, far from obvious, is that the unsuspected epithet ‘this’ is not quite properly applicable to, say, an electron, except with caution, in a restricted sense, and sometimes not at all.” E. Schrödinger ([35], p.197)

Schrödinger was in both cases specifically referring to what are called ‘indistinguishable particles’, the subject where these questions first arose. But it may be claimed that the whole formal structure of quantum mechanics is in conflict with the notion of individuals, as we will discuss below.

2.1 Quantum statistics: Bose-Einstein and Fermi-Dirac

It is commonly argued that elementary particles (electrons, quarks, neutrinos, and so on) are nomological objects: their characteristic properties are fixed by law. Particles of a given kind—for example, electrons—are by definition all exactly equal to one another. Inside a class, they are indistinguishable. Thus, following Leibniz, they would be all considered as *one and the same thing*. In such case, one would have to admit that there is only one electron in the whole world—something that is strongly denied by experiment [37].

Quantum theory prescribes a new way of counting electrons, photons and the like. Quanta obey Bose-Einstein and Fermi-Dirac statistics, in opposition to classical systems, which obey Maxwell-Boltzmann statistics.⁴ Consider two indistinguishable particles, for example electrons. If particles were assumed to possess individuality, they could be labeled as ‘electron 1’ and ‘electron 2’. Suppose they have to be arranged in two boxes. If they were objects, then they could actually be labeled because they may be at least distinguished by their positions—such as ‘being in the left box’ or ‘being in the right box’—there are four possibilities: both electrons in the left box, both in the right box, electron 1 in the left box and electron 2 in the right box and viceversa. These ‘labeled’ electrons would obey Maxwell-Boltzmann statistics and all four possibilities should be considered. Instead, quantum statistics impose that the two last options, because there is no way of distinguishing which electron is 1 and which one is 2, have to be considered as one and the same. If the particles were bosons, there would only be three possibilities left, while if they were fermions, there would be only one—because, in addition, the two first arrangements are prohibited for fermions by the Pauli exclusion principle. It is difficult to justify this way of counting the possible arrangements if particles are assumed to possess individuality and thus, the possibility of being labeled. This tension between individuality and quantum statistics is usually given as a justification for the fact that indistinguishable particles cannot be considered as individuals. However, the standard quantum formalism is constituted in the mathematics based on set theory. But how could quantum non-individuals be collected in sets when Cantor stated that sets may be regarded as “...collections of definite and separate objects of our intuition or our thought”? In order to face this serious inconsistency, there have been several proposed approaches to develop quantum set theories⁵ that acknowledge indistinguishability “right from the start”. In these frames, the principle of identity has restricted applicability.

There is, nevertheless, a standard way to deal with the subject of quantum statistics retaining particle indexation. This is done by restricting the states available for the particles by imposing that, if particles are indistinguishable, then they can only access symmetrized (with respect to particle interchange) states if they are bosons, or antisymmetrized states if they are fermions.

⁴See [17] for a very complete review.

⁵See for example [5], [21], [22]. There are also other quantum set theories [10], [15], [16], [26], [32], [36] that follow the suggestion made by von Neumann in relation to quantum logic.

All other states are prohibited or it is supposed that they are never realized in nature. This trick of first labeling and then restricting the available states allows to reproduce quantum statistics satisfactorily, without dropping particle indexation. But, as stated in [30], this procedure can be criticized, for one should still acknowledge why there appear in the formalism ‘prohibited states’ which may be thus considered as “surplus structure”. In fact, there has been considerable work around the existence of particles (‘paraparticles’) obeying other types of statistics (‘parastatistics’). In [30] it is argued that, as long as parastatistics are never observed in nature, then the usual procedure to construct the set of possible states —technically, the labeled tensor product Hilbert space formalism— should be replaced by another one that does not index particles, for example the Fock-space formalism. Though these formulations are not always equivalent,⁶ Fock-space formalism covers all experimental facts described by the standard formulation and satisfies the requirement that particles are not labeled —so that no surplus structure appears within it. It is important to remark however, that —as Steven French and Décio Krause argue in [17]— the formal construction of the Fock-space does use the standard set theoretical framework, which presupposes classical individuality on its foundations. This seems to be, thus, not a genuine solution.

The above outlined arguments show a deep blundering in the understanding of the meaning of the quantum formalism. On the one hand, experiments seem to rule out individuals but, on the other hand, the mathematics (set theory) used to formulate quantum mechanics presupposes the notion of individuality. Furthermore, we agree with Michael Readhead and Paul Teller when they say that:

“Interpreters of quantum mechanics largely agree that classical concepts do not apply without alteration or restriction to quantum objects. In Bohr’s formulation this means that one cannot simultaneously apply complementary concepts, such as position and momentum, without restriction. In particular, this means that one cannot attribute classical, well defined trajectories to quantum systems. But in a more fundamental respect it would seem that physicists, including Bohr, continue to think of quantum objects classically as individual things, capable, at least conceptually, of bearing labels. It is this presumption and its implications which we need to understand and critically examine.” M. Readhead and P. Teller ([30], p.202)

To deal with the indistinguishability of quanta we believe, along with Krause [31], that quantum set theories —which incorporate quantum non individuality without using particle labeling— deserve further investigation. We also believe that it is an interesting task to construct a Fock-space formalism using quaset theory, thus avoiding intermediate indexations and incorporating quantum indistinguishability “right from the start” [13].

2.2 Quantum individuals: Kochen-Specker theorem and superpositions

Up to now, we have discussed individuality and identity in the case of the statistical properties of indistinguishable quanta. We go now a step further and claim that the failure of the applicability of the notion of individuality occurs in a more general frame. Indeed, it occurs within the whole structure of quantum mechanics.

Let us consider the set \mathcal{L} of physical properties of a quantum system. The formalism of the theory associates to each physical magnitude a mathematical object —an operator, called “observable”, over the Hilbert space of states of the system— and the Heisenberg principle states that not all magnitudes may possess (accurate) values at the same time. This must not be interpreted as a consequence of our ignorance or of our inexact procedures to determine

⁶See, for example, [18].

them. Only subsets of ‘compatible’ magnitudes may simultaneously possess values. The indetermination of the values of incompatible pairs is a matter of principle. In fact, it is one of the fundamental physical principles from which the formal structure of the theory may be derived. In mathematical terms, observables linked by the Heisenberg principle do not commute and thus, physical magnitudes obey a non-commutative algebra —technically, the projectors in which they decompose are structured in a modular lattice in the finite case. This is strongly different from the classical realm —where they are structured in a Boolean lattice— and thus there exist (Boolean) valuations of all propositions about physical magnitudes. The different algebraic structure of the quantum properties has as its counterpart the different meaning of the logical connectives among propositions regarding properties. Thus, if we naively try to interpret them as classical properties, as properties ‘possessed by the system’, we are faced to all kind of no-go theorems that preclude this possibility. Most remarkably is the Kochen-Specker (KS) theorem [19] which explicitly shows the fact that within the formal structure of quantum mechanics, it is not possible to jointly assign truth values to different not-disjoint subsets of mutually compatible properties. This is a very strong impediment to get an image of quantum systems in some sense close to classical objects. Continuing with the investigation in [28, 29], we claim that the conclusion which must be driven from the KS theorem is that the quantum wave function cannot be conceived in terms of the state of *an individual which possesses properties*. The possible mathematical representations which expose the quantum wave function from different basis cannot be interpreted as related to properties which preexist simultaneously. Thus, the KS theorem shows the impossibility to unify the different representations in a unique and singular ‘whole’, in something which can be considered as *an individual*.

However, not even when a single basis—a particular mathematical representation to express the state of the system—is taken into account we can return so easily to the notion of individual without contradicting the formalism of the theory. In general, a quantum state (expanded in a chosen basis) is a linear combination of the elements of the basis. This is what is called a *superposition*. Paul Dirac called special attention with respect to its importance:

“The nature of the relationships which the superposition principle requires to exist between the states of any system is of a kind that cannot be explained in terms of familiar physical concepts. One cannot in the classical sense picture a system being partly in each of two states and see the equivalence of this to the system being completely in some other state. There is an entirely new idea involved, to which one must get accustomed and in terms of which one must proceed to build up an exact mathematical theory, without having any detailed classical picture.” P. Dirac ([8], p.12)

The idea of regarding a superposition as representing the state of an individual that possess properties does not work.⁷ A superposition is a pondered by complex numbers sum of various states of the system. Let us suppose that one of them is ‘up’ and the other is ‘down’ (or ‘dead’ and ‘alive’, as is the case in the famous example of poor Schrödinger’s cat [33]). We immediately recognize that both states ‘up’ and ‘down’ cannot be the simultaneous states of the same individual entity. The relationship between the states in the superposition is what Dirac points out as remaining far away from our familiar physical concepts.

According to the principle of contradiction—that which Aristotle, Leibniz and Kant considered as the most certain of all principles—everything *is* or *is not* the case. But, if we think of the components of the superposition as states which exist simultaneously, we are faced to a *contradiction*. To escape from it, attempts were made to interpret the superposition as a mathematical object expressing that the system is in one of its (unknown) state components. In which one of them, would be ‘discovered’ by measurement. Measurement would then reveal the

⁷See also [28] for discussion of the principle of superposition in relation to the notion of entity.

‘true’ state of the system. But we know this idea is simply wrong as has been shown through various theorems [25]. It is not possible to provide an *ignorance interpretation* of the elements of a superposition. The elements of the superposition do not exist in the mode of being of actuality, the only mode of being which we are accustomed to call *real*.

All these considerations confront ourselves, as in the case of indistinguishable quanta, with serious difficulties to interpret the formalism in terms of individual entities. If one firmly believes that such a thing as a *superposition* refers to something which has physical existence —and this is what quantum mechanics tells us if taken seriously— it seems we might be in the need of creating a new way of dealing with these elements of the quantum formalism, a way which should not depend on the classical idea of individual entity.

3 Superposition of the one and the many

In general, when physicists say that quantum systems have an *undefined* ‘particle number’, it may be understood that the number of particles varies in time because the system is open or because particles are created or destroyed. Thus, it may be presupposed that the state of affairs is such that a definite particle number can be attributed to the system even when one does not know it at every instant of time. In this frame, the Fock-space formalism is used because, within it, superpositions of states corresponding to different particle numbers are allowed. But the question arises of how many particles are encompassed in such a superposition state.

3.1 How many?

Performing a single measurement over a quantum system does not allow, as we have already discussed, to attribute the result of the measurement to a property which the system possessed before the measurement was performed without giving rise to serious problems. This is also the case when the said property is the ‘number of particles’ of the system. Suppose that the state is a superposition of two elementary states representing ‘two particles’ and ‘five particles’. The coefficients in the superposition, i.e. the numbers that ponder the elementary states in the sum, allow to predict the probability of obtaining two or five particles (no other result is allowed) when performing a measurement. Now, suppose that five particles are detected in a single measurement. We still cannot attribute this finding to the actual state before the process of measurement because the number of particles was not definite. The assertion that this is so because the number of particles may be varying in time for particles are being constantly created and destroyed, assumes that at each instant of time the number of particles is a determined *definite value*, a statement inconsistent with the formalism.

The conclusion that we are forced to derive from the formalism is that a system in a state that is a superposition of several defined particle number states has no cardinal. In other words, we claim that the particle number is *indefinite* in the same sense as any other property when we deal with a superposition. There are only few particular cases in which the cardinal can be considered to be definite valued —when the state is an eigenstate of the particle number operator— or in which we can attribute to ignorance the indetermination in the cardinal —when the state is a statistical mixture of defined number states. But, in the general case, the incapability of knowing the particle number does not come from our ignorance about the system but from the fact that the cardinal does not exist. It is important to point out one more time that the so called ‘particle number’ only appears, in general, after the measurement process is performed. And, as the result of a measurement cannot be attributed in general to a property pertaining to a system, ‘counting’ in quantum mechanics is qualitatively different from counting the quantity of elements of classical systems.

In which sense do we talk about quantum systems composed for example by a single photon? What do we mean when we use the words ‘single photon’? Experiments on quantum systems

sometimes show corpuscular features, and this suggests an idea of individuality. This idea is a base for developing the concept of ‘particle’ and subsequently, the notion of ‘particles aggregate’. But these are definite experimental arrangements which force the appearance of particle characteristics as a final result of a single process. Experiments are designed to find out which *is* the particle number, but this does not mean that the resulting number pertains to the system under study. On the contrary, it refers to the definite *process* which takes place in each measurement. We are not allowed to consider the system as an aggregate of individuals as if they were simple objects.

3.2 What would an adequate formalism be like?

How would the characteristics be of an ontology which is not founded on entities? It is hard to imagine it, but we can suspect that ‘to have a definite number’ would not necessarily need to be a principal characteristic of it. To have a definite number, is something to which entities are always tied. Thus, it would not be surprising that a Fock-space formulation of quantum mechanics based on quasi-set theory would be more adequate than the wave mechanics formulation. If we accept that physics refers in some sense to Being, this could be considered as an interesting example about how we could speak about Being without appealing to entities. Something which is not an individual entity, needs not to be *a one*, nor *a many*. The (historically constructed) notion of ‘number’ needs not to be applied to it. Therefore, that which is expressed through a superposition is not a one, nor a many, but notwithstanding, *it is*. In order to provide a formalism which comprises the mentioned features we have proposed an alternative procedure ([13], [14]) that resembles that of the Fock-space formalism but based on quaset theory \mathcal{Q} which genuinely avoids artificial labeling.

We briefly review first the main ideas of quasi-set theory \mathcal{Q} following mainly [23]. Intuitively speaking, \mathcal{Q} is obtained by applying ZFU-like (Zermelo-Fraenkel plus *Urelemente*) axioms to a basic domain composed of m -atoms (the new ingredients that stand for indistinguishable quanta, and to which the usual concept of identity does not apply), M -atoms and aggregates of them. The theory still admits a primitive concept of quasi-cardinal, which intuitively stands for the ‘quantity’ of objects in a collection. This is made so that certain quasi-sets x (in particular, those whose elements are q -objects) may have a quasi-cardinal, written $qc(x)$, but not an associated ordinal. It is also possible to define a translation from the language of ZFU into the language of \mathcal{Q} in such a way so that there is a ‘copy’ of ZFU in \mathcal{Q} (the ‘classical’ part of \mathcal{Q}). In this copy, all the usual mathematical concepts can be defined (inclusive the concept of ordinal for the \mathcal{Q} -sets, the copy of standard sets in \mathcal{Q}), and the \mathcal{Q} -sets turn out to be those quasi-sets whose transitive closure (this concept is like the usual one) does not contain m -atoms.

In \mathcal{Q} , ‘pure’ quasi-sets have only m -atoms as elements (although these elements may be not always indistinguishable from one another), and to them it is assumed that the usual notion of identity cannot be applied (the expressions $x = y$ and its negation, $x \neq y$, are not well formed formulas if either x or y stand for m -atoms). Notwithstanding, there is a primitive relation \equiv of indistinguishability having the properties of an equivalence relation, and a defined concept of *extensional identity*, not holding among m -atoms, which has the properties of standard identity of classical set theories. More precisely, we write $x =_E y$ (x and y are extensionally identical) iff they are both qsets having the same elements (that is, $\forall z(z \in x \longleftrightarrow z \in y)$) or they are both M -atoms and belong to the same qsets (that is, $\forall z(x \in z \longleftrightarrow y \in z)$). From now on, we shall use the symbol “=” for the extensional equality, except when explicitly mentioned.

Since the elements of a quasi-set may have properties (and satisfy certain formulas), they can be regarded as *indistinguishable* without turning to be *identical* (that is, being *the same object*), that is, $x \equiv y$ does not entail $x = y$. Since the relation of equality (and the concept of identity) does not apply to m -atoms, they can also be thought of as entities devoid of individuality. For

details about \mathcal{Q} and about its historical motivations, see [17, Chap. 7].

One of the main features of \mathcal{Q} is its ability to take into account in ‘set-theoretical terms’ the non observability of permutations in quantum physics, which is one of the most basic facts regarding indistinguishable quanta. In standard set theories, if $w \in x$, then of course $(x - \{w\}) \cup \{z\} = x$ iff $z = w$. That is, we can ‘exchange’ (without modifying the original arrangement) two elements iff they are *the same* element, by force of the axiom of extensionality. But in \mathcal{Q} there is a theorem guarantying the unobservability of permutations; in other words,

Theorem 3.1. *Let x be a finite quasi-set such that x does not contain all indistinguishable from z , where z is an m -atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists w' such that $(x - z') \cup w' \equiv x$*

Here z' and w' stand for a quasi-set with quasi-cardinal 1 whose only element is indistinguishable (but not identical) from z and w respectively.

We outline now the construction of the state space $V_{\mathcal{Q}}$ that respects indistinguishability in all steps working within \mathcal{Q} , mainly following [13] and [14]. Let us consider a quasi-set $\epsilon = \{\epsilon_i\}_{i \in I}$, where I is an arbitrary collection of indexes (this makes sense in the ‘classical part’ of \mathcal{Q}). We take the elements ϵ_i to represent the eigenvalues of a physical magnitude of interest. Consider then the quasi-functions f (this concept generalizes that of function), $f : \epsilon \longrightarrow \mathcal{F}_p$, where \mathcal{F}_p is the quasi-set formed of finite and pure quasi-sets. f is the quasi-set formed of ordered pairs $\langle \epsilon_i; x \rangle$ with $\epsilon_i \in \epsilon$ and $x \in \mathcal{F}_p$. Let us choose these quasi-functions in such a way that whenever $\langle \epsilon_{i_k}; x \rangle$ and $\langle \epsilon_{i_{k'}}; y \rangle$ belong to f and $k \neq k'$, then $x \cap y = \emptyset$. Let us further assume that the sum of the quasi-cardinals of the quasi-sets which appear in the image of each of these quasi-functions is finite, and then, $qc(x) = 0$ for every x in the image of f , except for a finite number of elements of ϵ . Let us call \mathcal{F} the quasi-set formed of these quasi-functions. If $\langle x; \epsilon_i \rangle$ is a pair of $f \in \mathcal{F}$, we will interpret that the energy level ϵ_i has occupation number $qc(x)$. These quasi-functions will be represented by symbols such as $f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_m}}$ (or by the same symbol with permuted indexes). This indicates that the levels $\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_m}$ are occupied. It will be taken as convention that if the symbol ϵ_{i_k} appears j -times, then the level ϵ_{i_k} has occupation number j . The levels that do not appear have occupation number zero.

It is important to point out that the order of the indexes in $f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}}$ has no meaning at all because up to now, there is no need to define any particular order in ϵ , the domain of the quasi-functions of \mathcal{F} . Nevertheless, we may define an order in the following way. For each quasi-function $f \in \mathcal{F}$, let $\{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_m}\}$ be the quasi-set formed by the elements of ϵ such that $\langle \epsilon_{i_k}, X \rangle \in f$ and $qc(X) \neq 0$ ($k = 1 \dots m$). We call $supp(f)$ this quasi-set (the *support* of f). Then consider the pair $\langle o, f \rangle$, where o is a bijective quasi-function $o : \{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_m}\} \longrightarrow \{1, 2, \dots, m\}$. Each of these quasi-functions o define an order on $supp(f)$. For each $f \in \mathcal{F}$, if $qc(supp(f)) = m$, then, there are $m!$ orderings. Then, let \mathcal{OF} be the quasi-set formed by all the pairs $\langle o, f \rangle$, where $f \in \mathcal{F}$ and o is a particular ordering in $supp(f)$. Thus, \mathcal{OF} is the quasi-set formed by all the quasi-functions of \mathcal{F} with ordered support. For this reason, $f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}} \in \mathcal{OF}$ refers to a quasifunction $f \in \mathcal{F}$ and a special ordering of $\{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}\}$. For the sake of simplicity, we will use the same notation as before. But now the order of the indexes is *meaningful*. It is also important to remark, that the order on the indexes must not be understood as a labeling of particles, for it easy to check, as above, that the permutation of particles does not give place to a new element of \mathcal{OF} . This is so because a permutation of particles operating on a pair $\langle o, f \rangle \in \mathcal{OF}$ will not change f , and so, will not alter the ordering. We will use the elements of \mathcal{OF} later, when we deal with fermions.

A linear space structure is required to adequately represent quantum states. To equip \mathcal{F} and \mathcal{OF} with such a structure, we need to define two operations “ \star ” and “ $+$ ”, a product by scalars and an addition of their elements, respectively. Call C the collection of quasi-functions

which assign to every $f \in \mathcal{F}$ (or $f \in \mathcal{OF}$) a complex number (again, built in the ‘classical part’ of \mathcal{Q}). That is, a quasi-function $c \in C$ is a collection of ordered pairs $\langle f; \lambda \rangle$, where $f \in \mathcal{F}$ (or $f \in \mathcal{OF}$) and λ a complex number. Let C_0 be the subset of C such that, if $c \in C_0$, then $c(f) = 0$ for almost every $f \in \mathcal{OF}$ (i.e., $c(f) = 0$ for every $f \in \mathcal{OF}$ except for a finite number of quasi-functions). We can define in C_0 a sum and a product by scalars in the same way as it is usually done with functions as follows:

Definition 3.2. Let α, β and $\gamma \in \mathcal{C}$, and c, c_1 and c_2 be quasi-functions of C_0 , then

$$(\gamma * c)(f) := \gamma(c(f)) \quad \text{and} \quad (c_1 + c_2)(f) := c_1(f) + c_2(f)$$

The quasi-function $c_0 \in C_0$ such that $c_0(f) = 0$, for any $f \in \mathcal{F}$, acts as the null element of the sum, for $(c_0 + c)(f) = c_0(f) + c(f) = 0 + c(f) = c(f), \forall f$. With the sum and the multiplication by scalars defined above we have that $(C_0, +, *)$ is a complex vector space. Each one of the quasi-functions of C_0 should be interpreted in the following way: if $c \in C_0$ (and $c \neq c_0$), let $f_1, f_2, f_3, \dots, f_n$ be the only functions of C_0 which satisfy $c(f_i) \neq 0$ ($i = 1, \dots, n$). These quasi-functions exist because, as we have said above, the quasi-functions of C_0 are zero except for a finite number of quasi-functions of \mathcal{F} . If λ_i are complex numbers which satisfy that $c(f_i) = \lambda_i$ ($i = 1, \dots, n$), we will make the association $c \approx (\lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_n f_n)$. The symbol \approx must be understood in the sense that we use this notation to represent the quasi-function c . The idea is that the quasi-function c represents the pure state which is a linear combination of the states represented by the quasi-functions f_i according to the interpretation given above.

In order to calculate probabilities and mean values, we have to introduce a scalar product, in fact two of them: \circ for bosons and \bullet for fermions, thus obtaining two (normed) vector spaces (V_Q, \circ) and (V_Q, \bullet) :

Definition 3.3. Let δ_{ij} be the Kronecker symbol and $f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}}$ and $f_{\epsilon_{i'_1} \epsilon_{i'_2} \dots \epsilon_{i'_m}}$ two basis vectors, then

$$f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}} \circ f_{\epsilon_{i'_1} \epsilon_{i'_2} \dots \epsilon_{i'_m}} := \delta_{nm} \sum_p \delta_{i_1 p i'_1} \delta_{i_2 p i'_2} \dots \delta_{i_n p i'_n}$$

The sum is extended over all the permutations of the set $i' = (i'_1, i'_2, \dots, i'_n)$ and for each permutation p , $pi' = (pi'_1, pi'_2, \dots, pi'_n)$.

This product can be easily extended over linear combinations.

Definition 3.4. Let δ_{ij} be the Kronecker symbol, $f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}}$ and $f_{\epsilon_{i'_1} \epsilon_{i'_2} \dots \epsilon_{i'_m}}$ two basis vectors, then

$$f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}} \bullet f_{\epsilon_{i'_1} \epsilon_{i'_2} \dots \epsilon_{i'_m}} := \delta_{nm} \sum_p \sigma_p \delta_{i_1 p i'_1} \delta_{i_2 p i'_2} \dots \delta_{i_n p i'_n}$$

where: $s^p = +1$ if p is even and $s^p = -1$ if p is odd.

The result of this second product \bullet is an antisymmetric sum of the indexes which appear in the quasi-functions. In order that the product is well defined, the quasi-functions must belong to \mathcal{OF} . Once this product is defined over the basis functions, we can extend it to linear combinations, in a similar way as for bosons. If the occupation number of a product is more or equal than two, then the vector has null norm. As it is a vector of null norm, the product of this vector with any other vector of the space would yield zero, and thus the probability of observing a system in a state like it vanishes. This means that we can add to any physical state an arbitrary linear combination of null norm vectors for they do not contribute to the scalar product, which is the meaningful quantity.

With these tools and using the language of \mathcal{Q} , the formalism of QM may be completely rewritten giving a straightforward answer to the problem of giving a formulation of QM in which intrinsic indistinguishability is taken into account from the beginning, without artificially introducing extra postulates. We make the following association in order to turn the notation similar to that of the standard formalism. For each quasi-function $f_{\epsilon_{i_1}\epsilon_{i_2}\dots\epsilon_{i_n}}$ of the quasi-sets \mathcal{F} or \mathcal{OF} constructed above, we will write $\alpha f_{\epsilon_{i_1}\epsilon_{i_2}\dots\epsilon_{i_n}} := \alpha|\epsilon_{i_1}\epsilon_{i_2}\dots\epsilon_{i_n})$ with the obvious corresponding generalization for linear combinations. Once normalized to unity, the states constructed using \mathcal{Q} , are equivalent to the symmetrized vectors for bosonic states and we have shown that commutation relations equivalent to the usual ones hold, thus being both formulations equivalent for bosons.

For fermions, there are some subtleties involved in the construction. First of all, let us recall the action of the creation operator c_α^\dagger : let ζ represent a collection of indexes with non null occupation number, then $C_\alpha^\dagger|\zeta) = |\alpha\zeta)$. If α was already in the collection ζ , then $|\alpha\zeta)$ is a vector with null norm. As said above, to have null norm implies that $(\psi|\alpha\zeta) = 0$ for all $|\psi)$. Moreover, if a linear combination of null norm vectors were added to the vector representing the state of a system, this addition would not give place to observable results because the terms of null norm do not contribute to the mean values or to the probabilities. In order to express this situation, we define the following relation:

Definition 3.5. *Two vectors $|\varphi)$ and $|\psi)$ are similar (and we will write $|\varphi) \cong |\psi)$) if the difference between them is a linear combination of null norm vectors.*

With all of this, it is straightforward to demonstrate the equivalence of the anticommutation relations in V_Q and in the standard Fock-space. Thus, we can conclude that both formulations are equivalent also for fermions. To avoid particle labeling in the expressions for observables, in Fock-space formalism they are written in terms of creation and annihilation operators.

4 Modal interpretations of quantum mechanics

We have argued against forcing the interpretation of the quantum formalism in terms of actual individuals. In view of the difficulties posed by the theory to give place to actual entities, modal interpretations of QM attempt to consider the role of *possible properties* in the orthodox quantum formalism giving place to a consistent discourse about *possible entities*.

Modal interpretations have continued the footprints left by Niels Bohr, Max Born, Werner Heisenberg and Wolfgang Pauli and continued the path on the lines drawn by Bas van Fraassen, Simon Kochen, Dennis Dieks and many others, searching for the different possibilities of interpreting the formalism of the theory.⁸ Modal interpretations may be thought to be a study of the constraints under which one is able to talk a consistent classical discourse without contradiction with the quantum formalism. Following the general outline provided in [11] one might state in general terms that a modal interpretation is best characterized by the following points:

1. One of the most significant features of modal interpretations is that they *stay close to the standard formulation*. Following van Fraassen's recommendation, one needs to learn from the formal structure of the theory in order to develop an interpretation. This is different from many attempts which presuppose an ontology and then try to fit it into the formalism.
2. Modal interpretations are *non-collapse interpretations*. The evolution is always given by the Schrödinger equation of motion and the collapse of the wave function is nothing but the path from the possible to the actual, it is not considered a physical process.

⁸ See for example [6, 20, 38, 39].

3. Modal interpretations *ascribe possible properties to quantum systems*. The property ascription depends on the states of the systems and applies regardless of whether or not measurements are performed. There is a distinction between the level of possibility and that of actuality which are related through an interpretational rule. (Technically, in addition to actual properties interpreted in the orthomodular lattice \mathcal{L} of quantum logic, there is a set of possible properties $\diamond\mathcal{L}$ which may be regarded as constituting the center of the enriched lattice.)
4. Modality *is not interpreted in terms of ignorance*. There is no ignorance interpretation of the probability distribution assigned to the physical properties. The state of the system determines all there is to know. For modal interpretations there is no such thing as ‘hidden variables’ from which we could get more information. One can formulate a KS theorem for modalities which expresses the irreducible contextual character of quantum mechanics even in the case of enriching its language with a modal operator.

4.1 ‘Possible’ individuals?

Modal interpretations intend to discuss about systems with properties going beyond the instrumentalist positions which only talk about measurement outcomes. However, it is not obvious which properties can be considered as possessing definite values. In particular there are several no-go theorems which restrict this possibility [1, 41].

Still today there is a tension in modal interpretations which has not been solved. Although Bacciagaluppi claims that “despite the name, the modal interpretation in the version of Vermaas and Dieks is a theory about actualities” ([2], p.74), Dieks still seems to present a different position:

“[...] according to modal interpretations the quantum formalism does not tell us what actually is the case in the physical world, but rather provides us with a list of possibilities and their probabilities. The modal viewpoint is therefore that quantum theory is about what may be the case, in philosophical jargon, quantum theory is about modalities.” D. Dieks ([7])

It might be thought that such interpretation of quantum mechanics in terms of modalities opens the door to a new way to refer to individuals, not in terms of actuality but rather in terms of possibilities. Thus, one might speak about ‘possible individuals’ instead of ‘actual individuals’. At first sight, one might think it is still achievable to recover the notion of individuality and identity in quantum mechanics if one is careful enough to remain within the realm of possibility. But, once again, the quantum formalism is ready to stop any move which intends to constrain its interpretation within the notion of individuality. In fact, a theorem has been developed which precludes to consistently interpret the formalism in terms of ‘possible individuals’.

4.2 Limits of modality: the MKS theorem

In order to stay away from inconsistencies when speaking about properties which pertain to the system, one must acknowledge the limitations imposed by the KS theorem. To do so, modal interpretations assign to the system only a set of definite properties. This is not achievable when talking about properties which pertain to different contexts (see for discussion [3]).

At first sight it might seem paradoxical that, even though modal interpretations of quantum mechanics talk about modalities, KS theorem refers to actual values of physical properties. Elsewhere, and following the line of thought of quantum logic, we have investigated the question whether KS theorem has something to say about possibility and its relation to actuality [9, 12]. The answer was provided via a characterization of the relations between actual and possible

properties pertaining to different contexts. By applying algebraic and topological tools we studied the structure of the orthomodular lattice of actual propositions enriched with modal propositions. Let us briefly recall the results. As usual, given a proposition about the system, it is possible to define a context from which one can predicate with certainty about it (and about a set of propositions that are compatible with it) and predicate probabilities about the other ones. This is to say that one may predicate truth or falsity of all possibilities at the same time, i.e. possibilities allow an interpretation in a classical system of propositions. In order to describe rigorously the formalism which allows to capture all propositions in a single structure, let \mathcal{L} be an orthomodular lattice. Given a, b, c in L , we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D$, $(a, b, c)D^*$ hold for all permutations of a, b, c . An element $z \in \mathcal{L}$ is called *central* iff for all elements $a, b \in L$ we have $(a, b, z)T$. We denote by $Z(\mathcal{L})$ the set of all central elements of \mathcal{L} and it is called the *center* of \mathcal{L} . $Z(\mathcal{L})$ is a Boolean sublattice of L [24, Theorem 4.15].

Let P be a proposition about a system and consider it as an element of an orthomodular lattice \mathcal{L} . If we refer with $\Diamond P$ to the possibility of P , then $\Diamond P$ will be a central element of \mathcal{L} . This interpretation of the possibility in terms of the Boolean algebra of central elements of \mathcal{L} reflects the fact that one can simultaneously predicate about all possibilities. This is so because it is always possible to establish Boolean homomorphisms of the form $v : Z(\mathcal{L}) \rightarrow \mathbf{2}$. Therefore, the key idea is to expand the orthomodular structure in such a way to include propositions about possibility. This expansion is performed in the following way: If P is a proposition about the system and P occurs, then it is trivially possible that P occurs. This is expressed as $P \leq \Diamond P$. In fact, to assume an actual property and a complete set of properties that are compatible with it determines a context in which the classical discourse holds. Classical consequences that are compatible with it, for example probability assignments to the actuality of other propositions, shear the classical frame. These consequences are the same ones as those which would be obtained by considering the original actual property as a possible property. This is interpreted as, if P is a property of the system, $\Diamond P$ is the smallest central element greater than P . With these tools, we are able to give an extension of the orthomodular structure by adding a possibility operator that fulfills the mentioned requirements. More precisely, the extension is a class of algebras, called Boolean saturated orthomodular lattices, that admits the orthomodular structure as a reduct and we demonstrate that they are a variety, i.e., definable by equations.

If \mathcal{L} is an orthomodular lattice and \mathcal{L}^\Diamond a Boolean saturated orthomodular one such that \mathcal{L} can be embedded in \mathcal{L}^\Diamond , we say that \mathcal{L}^\Diamond is a modal extension of \mathcal{L} . Given \mathcal{L} and a modal extension \mathcal{L}^\Diamond , we define the *possibility space* as the subalgebra of \mathcal{L}^\Diamond generated by $\{\Diamond P : P \in \mathcal{L}\}$. We denote by $\Diamond \mathcal{L}$ this space and it may be proved that it is a Boolean subalgebra of the modal extension. The possibility space represents the modal content added to the discourse about properties of the system.

Within this frame, the actualization of a possible property acquires a rigorous meaning. Let \mathcal{L} be an orthomodular lattice, $(W_i)_{i \in I}$ the family of Boolean sublattices of \mathcal{L} and \mathcal{L}^\Diamond a modal extension of \mathcal{L} . If $f : \Diamond \mathcal{L} \rightarrow \mathbf{2}$ is a Boolean homomorphism, an actualization compatible with f is a global valuation $(v_i : W_i \rightarrow \mathbf{2})_{i \in I}$ such that $v_i \upharpoonright W_i \cap \Diamond \mathcal{L} = f \upharpoonright W_i \cap \Diamond \mathcal{L}$ for each $i \in I$.

Compatible actualizations represent the passage from possibility to actuality. When taking into account compatible actualizations from different contexts, the following KS theorem for modalities can be proved [9]:

Theorem 4.1. *Let \mathcal{L} be an orthomodular lattice. Then \mathcal{L} admits a global valuation iff for each possibility space there exists a Boolean homomorphism $f : \Diamond \mathcal{L} \rightarrow \mathbf{2}$ that admits a compatible actualization.* \square

The modal KS (MKS) theorem shows that no enrichment of the language about actual properties results in something close to a classical image. The conclusion which can be derived from the MKS theorem is that the formalism of quantum mechanics does not only deny the

possibility of talking about an ‘actual entity’, but even the term ‘possible entity’ remains a meaningless notion within its domain of discourse.

5 Departing from Plato’s footnotes

Nietzsche and Heidegger claim that the tradition initiated by Aristotle and followed by the rest of the subsequent occidental philosophers of thinking Being in terms of entities needs to be criticized. From the landscape of quantum mechanics we may argue that all investigations point in the direction of its incompatibility with the notion of entity. First, a quantum system cannot be thought in terms of the unity of its properties because there always exist incompatible properties and besides, it is inconsistent to say that properties pertain to it, not even those measured properties. This is also the fact when the brought up property is the number of particles in the system. Furthermore, it is not allowed to consider ‘possible entities’, because they fall under analogous criticisms to those of actual ones. Last in our series of examples, indistinguishable quanta show similar lacking of individuality when being considered in a collection, which cause a different statistical behavior than that of classical aggregates. Perhaps, this is the time in physics in which we need to abandon the Aristotelian tradition of thinking the world in terms of entities, in order to make a cogent picture of quantum phenomena.

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